

# THE TENSOR PRODUCT OF GORENSTEIN-PROJECTIVE MODULES OVER CATEGORY ALGEBRAS

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**ABSTRACT.** For a finite free and projective EI category, we prove that Gorenstein-projective modules over its category algebra are closed under the tensor product if and only if each morphism in the given category is a monomorphism.

## 1. INTRODUCTION

Let  $k$  be a field. Let  $\mathcal{C}$  be a finite category, that is, it has only finitely many morphisms, and consequently it has only finitely many objects. Denote by  $k\text{-mod}$  the category of finite dimensional  $k$ -vector spaces and  $(k\text{-mod})^{\mathcal{C}}$  the category of covariant functors from  $\mathcal{C}$  to  $k\text{-mod}$ .

Recall that the category  $k\mathcal{C}\text{-mod}$  of left modules over the category algebra  $k\mathcal{C}$  is identified with  $(k\text{-mod})^{\mathcal{C}}$ ; see [7, Proposition 2.1]. The category  $k\mathcal{C}\text{-mod}$  is a symmetric monoidal category. More precisely, the tensor product  $-\hat{\otimes}-$  is defined by

$$(M\hat{\otimes}N)(x) = M(x) \otimes_k N(x)$$

for any  $M, N \in (k\text{-mod})^{\mathcal{C}}$  and  $x \in \text{Obj}\mathcal{C}$ , and  $\alpha.(m \otimes n) = \alpha.m \otimes \alpha.n$  for any  $\alpha \in \text{Mor}\mathcal{C}$ ,  $m \in M(x)$ ,  $n \in N(x)$ ; see [8, 9].

Let  $\mathcal{C}$  be a finite EI category. Here, the EI condition means that all endomorphisms in  $\mathcal{C}$  are isomorphisms. In particular,  $\text{Hom}_{\mathcal{C}}(x, x) = \text{Aut}_{\mathcal{C}}(x)$  is a finite group for each object  $x$ . Denote by  $k\text{Aut}_{\mathcal{C}}(x)$  the group algebra. Recall that a finite EI category  $\mathcal{C}$  is *projective over  $k$*  if each  $k\text{Aut}_{\mathcal{C}}(y)$ - $k\text{Aut}_{\mathcal{C}}(x)$ -bimodule  $k\text{Hom}_{\mathcal{C}}(x, y)$  is projective on both sides; see [5, Definition 4.2].

Denote by  $k\mathcal{C}\text{-Gproj}$  the full subcategory of  $k\mathcal{C}\text{-mod}$  consisting of Gorenstein-projective  $k\mathcal{C}$ -modules. We say that  $\mathcal{C}$  is *GPT-closed*, if  $X, Y \in k\mathcal{C}\text{-Gproj}$  implies  $X\hat{\otimes}Y \in k\mathcal{C}\text{-Gproj}$ .

Let us explain the motivation to study GPT-closed categories. Recall from [6] that for a finite projective EI category  $\mathcal{C}$ , the stable category  $k\mathcal{C}\text{-Gproj}$  modulo projective modules has a natural tensor triangulated structure such that it is tensor triangle equivalent to the singularity category of  $k\mathcal{C}$ . In general, its tensor product is not explicitly given. However, if  $\mathcal{C}$  is GPT-closed, then the tensor product  $-\hat{\otimes}-$  on Gorenstein-projective modules induces the one on  $k\mathcal{C}\text{-Gproj}$ ; see Proposition 3.4. In this case, we have a better understanding of the tensor triangulated category  $k\mathcal{C}\text{-Gproj}$ .

**Proposition 1.1.** *Let  $\mathcal{C}$  be a finite projective EI category. Assume that  $\mathcal{C}$  is GPT-closed. Then each morphism in  $\mathcal{C}$  is a monomorphism.*

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The concept of a finite *free* EI category is introduced in [3].

**Theorem 1.2.** *Let  $\mathcal{C}$  be a finite projective and free EI category. Then the category  $\mathcal{C}$  is GPT-closed if and only if each morphism in  $\mathcal{C}$  is a monomorphism.*

## 2. GORENSTEIN TRIANGULAR MATRIX ALGEBRAS

In this section, we recall some necessary preliminaries on triangular matrix algebras and Gorenstein-projective modules.

Recall an  $n \times n$  upper triangular matrix algebra  $\Gamma = \begin{pmatrix} R_1 & M_{12} & \cdots & M_{1n} \\ & R_2 & \cdots & M_{2n} \\ & & \ddots & \vdots \\ & & & R_n \end{pmatrix}$ , where each  $R_i$  is an algebra for  $1 \leq i \leq n$ , each  $M_{ij}$  is an  $R_i$ - $R_j$ -bimodule for  $1 \leq i < j \leq n$ , and the matrix algebra map is denoted by  $\psi_{ilj} : M_{il} \otimes_{R_l} M_{lj} \rightarrow M_{ij}$  for  $1 \leq i < l < j < t \leq n$ ; see [5].

Recall that a left  $\Gamma$ -module  $X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$  is described by a column vector, where each  $X_i$  is a left  $R_i$ -module for  $1 \leq i \leq n$ , and the left  $\Gamma$ -module structure map is denoted by  $\varphi_{jl} : M_{jl} \otimes_{R_l} X_l \rightarrow X_j$  for  $1 \leq j < l \leq n$ ; see [5]. Dually, we have the description of right  $\Gamma$ -modules via row vectors.

**Notation 2.1.** *Let  $\Gamma_t$  be the algebra given by the  $t \times t$  leading principal submatrix of  $\Gamma$  and  $\Gamma'_{n-t}$  be the algebra given by cutting the first  $t$  rows and the first  $t$  columns*

*of  $\Gamma$ . Denote the left  $\Gamma_t$ -module  $\begin{pmatrix} M_{1,t+1} \\ \vdots \\ M_{t,t+1} \end{pmatrix}$  by  $M_t^*$  and the right  $\Gamma'_{n-t}$ -module  $(M_{t,t+1}, \dots, M_{tn})$  by  $M_t^{**}$ , for  $1 \leq t \leq n-1$ . Denote by  $\Gamma_t^D = \text{diag}(R_1, \dots, R_t)$  the sub-algebra of  $\Gamma_t$  consisting of diagonal matrices, and  $\Gamma'_{D,n-t} = \text{diag}(R_{t+1}, \dots, R_n)$  the sub-algebra of  $\Gamma'_{n-t}$  consisting of diagonal matrices.*

Let  $\Gamma = \begin{pmatrix} R_1 & M_{12} \\ 0 & R_2 \end{pmatrix}$  be an upper triangular matrix algebra. Recall that the  $R_1$ - $R_2$ -bimodule  $M_{12}$  is *compatible*, if the following two conditions hold; see [10, Definition 1.1]:

- (C1) If  $Q^\bullet$  is an exact sequence of projective  $R_2$ -modules, then  $M_{12} \otimes_{R_2} Q^\bullet$  is exact;
- (C2) If  $P^\bullet$  is a complete  $R_1$ -projective resolution, then  $\text{Hom}_{R_1}(P^\bullet, M_{12})$  is exact.

We use  $\text{pd}$  and  $\text{id}$  to denote the projective dimension and the injective dimension of a module, respectively.

**Lemma 2.2.** *Let  $\Gamma$  be an upper triangular matrix algebra satisfying all  $R_i$  Gorenstein. If  $\Gamma$  is Gorenstein, then each  $\Gamma_t$ - $R_{t+1}$ -bimodule  $M_t^*$  is compatible and each  $R_t$ - $\Gamma'_{n-t}$ -bimodule  $M_t^{**}$  is compatible for  $1 \leq t \leq n-1$ .*

*Proof.* Let  $\Lambda = \begin{pmatrix} S_1 & N_{12} \\ 0 & S_2 \end{pmatrix}$  be an upper triangular matrix algebra. Recall the fact that if  $\text{pd}_{S_1}(N_{12}) < \infty$  and  $\text{pd}(N_{12})_{S_2} < \infty$ , then  $N_{12}$  is compatible; see [10, Proposition 1.3]. Recall that  $\Gamma$  is Gorenstein if and only if all bimodules  $M_{ij}$

are finitely generated and have finite projective dimension on both sides; see [5, Proposition 3.4]. Then we have  $\text{pd}_{R_t}(M_t^{**}) < \infty$  and  $\text{pd}(M_t^*)_{R_{t+1}} < \infty$  for  $1 \leq t \leq n-1$ . By [5, Lemma 3.1], we have  $\text{pd}(M_t^{**})_{\Gamma'_{n-t}} < \infty$  and  $\text{pd}_{\Gamma_t}(M_t^*) < \infty$  for  $1 \leq t \leq n-1$ . Then we are done.  $\square$

Let  $A$  be a finite dimensional algebra over a field  $k$ . Denote by  $A\text{-mod}$  the category of finite dimensional left  $A$ -modules. The opposite algebra of  $A$  is denoted by  $A^{\text{op}}$ . We identify right  $A$ -modules with left  $A^{\text{op}}$ -modules.

Denote by  $(-)^*$  the contravariant functor  $\text{Hom}_A(-, A)$  or  $\text{Hom}_{A^{\text{op}}}(-, A)$ . Let  $X$  be a left  $A$ -module. Then  $X^*$  is a right  $A$ -module and  $X^{**}$  is a left  $A$ -module. There is an evaluation map  $\text{ev}_X : X \rightarrow X^{**}$  given by  $\text{ev}_X(x)(f) = f(x)$  for  $x \in X$  and  $f \in X^*$ . Recall that an  $A$ -module  $G$  is *Gorenstein-projective* provided that  $\text{Ext}_A^i(G, A) = 0 = \text{Ext}_{A^{\text{op}}}^i(G^*, A)$  for  $i \geq 1$  and the evaluation map  $\text{ev}_G$  is bijective; see [1, Proposition 3.8].

The algebra  $A$  is *Gorenstein* if  $\text{id}_A A < \infty$  and  $\text{id}_{A^{\text{op}}} A < \infty$ . It is well known that for a Gorenstein algebra  $A$  we have  $\text{id}_A A = \text{id}_{A^{\text{op}}} A$ ; see [11, Lemma A]. For  $m \geq 0$ , a Gorenstein algebra  $A$  is *m-Gorenstein* if  $\text{id}_A A = \text{id}_{A^{\text{op}}} A \leq m$ . Denote by  $A\text{-Gproj}$  the full subcategory of  $A\text{-mod}$  consisting of Gorenstein-projective  $A$ -modules, and  $A\text{-proj}$  the full subcategory of  $A\text{-mod}$  consisting of projective  $A$ -modules.

The following lemma is well known; see [1, Propositions 3.8 and 4.12 and Theorem 3.13].

**Lemma 2.3.** *Let  $m \geq 0$ . Let  $A$  be an  $m$ -Gorenstein algebra. Then we have the following statements.*

- (1) *An  $A$ -module  $M \in A\text{-Gproj}$  if and only if  $\text{Ext}_A^i(M, A) = 0$  for all  $i > 0$ .*
- (2) *If  $M \in A\text{-Gproj}$  and  $L$  is a right  $A$ -module with finite projective dimension, then  $\text{Tor}_i^A(L, M) = 0$  for all  $i > 0$ .*
- (3) *If there is an exact sequence  $0 \rightarrow M \rightarrow G_0 \rightarrow G_1 \rightarrow \cdots \rightarrow G_m$  with  $G_i$  Gorenstein-projective, then  $M \in A\text{-Gproj}$ .*  $\square$

**Lemma 2.4.** [10, Theorem 1.4] *Let  $M_{12}$  be a compatible  $R_1$ - $R_2$ -bimodule, and  $\Gamma = \begin{pmatrix} R_1 & M_{12} \\ 0 & R_2 \end{pmatrix}$ . Then  $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in \Gamma\text{-Gproj}$  if and only if  $\varphi_{12} : M_{12} \otimes_{R_2} X_2 \rightarrow X_1$  is an injective  $R_1$ -map,  $\text{Coker } \varphi_{12} \in R_1\text{-Gproj}$ , and  $X_2 \in R_2\text{-Gproj}$ .*  $\square$

We have a slight generalization of Lemma 2.4.

**Lemma 2.5.** *Let  $\Gamma$  be a Gorenstein upper triangular matrix algebra with each*

*$R_i$  a group algebra. Then  $X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \in \Gamma\text{-Gproj}$  if and only if each  $R_t$ -map*

$$\varphi_t^{**} = \sum_{j=t+1}^n \varphi_{tj} : M_t^{**} \otimes_{\Gamma'_{n-t}} \begin{pmatrix} X_{t+1} \\ \vdots \\ X_n \end{pmatrix} \rightarrow X_t \text{ sending } (m_{t,t+1}, \dots, m_{tn}) \otimes \begin{pmatrix} x_{t+1} \\ \vdots \\ x_n \end{pmatrix}$$

*to  $\sum_{j=t+1}^n \varphi_{tj}(m_{tj} \otimes x_j)$  is injective for  $1 \leq t \leq n-1$ .*

*Proof.* We have that each  $R_t\text{-}\Gamma'_{n-t}$ -bimodule  $M_t^{**}$  is compatible for  $1 \leq t \leq n-1$  by Lemma 2.2.

For the “only if” part, we use induction on  $n$ . The case  $n = 2$  is due to Lemma 2.4. Assume that  $n > 2$ . Write  $\Gamma = \begin{pmatrix} R_1 & M_1^{**} \\ 0 & \Gamma'_{n-1} \end{pmatrix}$ , and  $X = \begin{pmatrix} X_1 \\ X' \end{pmatrix}$ . Since  $X \in \Gamma$ -Gproj, by Lemma 2.4, we have that the  $R_1$ -map  $\varphi_1^{**} : M_1^{**} \otimes_{\Gamma'_{n-1}} X' \rightarrow X_1$  is injective and  $X' \in \Gamma'_{n-1}$ -Gproj. By induction, we have that each  $R_t$ -map  $\varphi_t^{**} : M_t^{**} \otimes_{\Gamma'_{n-t}} \begin{pmatrix} X_{t+1} \\ \vdots \\ X_n \end{pmatrix} \rightarrow X_t$  is injective for  $1 \leq t \leq n-1$ .

For the “if” part, we use induction on  $n$ . The case  $n = 2$  is due to Lemma 2.4. Assume that  $n > 2$ . Write  $\Gamma = \begin{pmatrix} R_1 & M_1^{**} \\ 0 & \Gamma'_{n-1} \end{pmatrix}$ , and  $X = \begin{pmatrix} X_1 \\ X' \end{pmatrix}$ . By induction, we have  $X' \in \Gamma'_{n-1}$ -Gproj. Since the  $R_1$ -map  $\varphi_1^{**} : M_1^{**} \otimes_{\Gamma'_{n-1}} X' \rightarrow X_1$  is injective and its cokernel belongs to  $R_1$ -Gproj for  $R_1$  a group algebra, we have  $X \in \Gamma$ -Gproj by Lemma 2.4.  $\square$

**Corollary 2.6.** *Let  $\Gamma$  be a Gorenstein upper triangular matrix algebra with each*

*$R_i$  a group algebra. Assume that  $X = \begin{pmatrix} X_1 \\ \vdots \\ X_s \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \Gamma$ -Gproj. Then each  $R_i$ -map  $\varphi_{is} : M_{is} \otimes_{R_s} X_s \rightarrow X_i$  is injective for  $1 \leq i < s \leq n$ .*

*Proof.* We write  $X = \begin{pmatrix} X' \\ X'' \end{pmatrix}$ , where  $X'' = \begin{pmatrix} X_{i+1} \\ \vdots \\ X_s \\ \vdots \\ 0 \end{pmatrix}$  for each  $1 \leq i < s \leq n$ . We

claim that each  $R_i$ -map  $f_{is} : M_{is} \otimes_{R_s} X_s \rightarrow M_i^{**} \otimes_{\Gamma'_{n-i}} X''$  sending  $m_{is} \otimes x_s$  to  $(0, \dots, m_{is}, \dots, 0) \otimes (0, \dots, x_s, \dots, 0)^t$  is injective, where  $(-)^t$  is the transpose. Since  $\varphi_{is} = \varphi_i^{**} \circ f_{is}$  for  $1 \leq i < s \leq n$ , then we are done by Lemma 2.5.

For the claim, we observe that for each  $1 \leq i < s \leq n$ , the  $R_i$ -map  $f_{is}$  is a composition of the following

$$M_{is} \otimes_{R_s} X_s \xrightarrow{g_{is}} (0, \dots, 0, M_{is}, \dots, M_{in}) \otimes_{\Gamma'_{n-i}} X'' \xrightarrow{\iota \otimes \text{Id}} M_i^{**} \otimes_{\Gamma'_{n-i}} X'',$$

where the right  $\Gamma'_{n-i}$ -map  $(0, \dots, M_{is}, \dots, M_{in}) \xrightarrow{\iota} M_i^{**}$  is the inclusion map and  $g_{is}$  sends  $m_{is} \otimes x_s$  to  $(0, \dots, m_{is}, \dots, 0) \otimes (0, \dots, x_s, \dots, 0)^t$ . We observe a  $R_i$ -map  $(0, \dots, M_{is}, \dots, M_{in}) \otimes_{\Gamma'_{n-i}} X'' \xrightarrow{g'_{is}} M_{is} \otimes_{R_s} X_s, (0, \dots, m_{is}, \dots, m_{in}) \otimes (0, \dots, x_s, \dots, 0)^t \mapsto m_{is} \otimes x_s$  satisfying  $g'_{is} \circ g_{is} = \text{Id}_{M_{is} \otimes_{R_s} X_s}$ . Hence the  $R_i$ -map  $g_{is}$  is injective. We observe that the right  $\Gamma'_{n-i}$ -modules  $(0, \dots, M_{is}, \dots, M_{in})$  and  $M_i^{**}$  have finite projective dimensions; see [5, Lemma 3.1], and  $X'' \in \Gamma'_{n-i}$ -Gproj by Lemma 2.4. Then the  $R_i$ -map  $\iota \otimes \text{Id}$  is injective by Lemma 2.3 (2).  $\square$

## 3. PROOF OF PROPOSITION 1.1

Let  $k$  be a field. Let  $\mathcal{C}$  be a finite category, that is, it has only finitely many morphisms, and consequently it has only finitely many objects. Denote by  $\text{Mor}\mathcal{C}$  the finite set of all morphisms in  $\mathcal{C}$ . The *category algebra*  $k\mathcal{C}$  of  $\mathcal{C}$  is defined as follows:  $k\mathcal{C} = \bigoplus_{\alpha \in \text{Mor}\mathcal{C}} k\alpha$  as a  $k$ -vector space and the product  $*$  is given by the rule

$$\alpha * \beta = \begin{cases} \alpha \circ \beta, & \text{if } \alpha \text{ and } \beta \text{ can be composed in } \mathcal{C}; \\ 0, & \text{otherwise.} \end{cases}$$

The unit is given by  $1_{k\mathcal{C}} = \sum_{x \in \text{Obj}\mathcal{C}} \text{Id}_x$ , where  $\text{Id}_x$  is the identity endomorphism of an object  $x$  in  $\mathcal{C}$ .

If  $\mathcal{C}$  and  $\mathcal{D}$  are two equivalent finite categories, then  $k\mathcal{C}$  and  $k\mathcal{D}$  are Morita equivalent; see [7, Proposition 2.2]. In particular,  $k\mathcal{C}$  is Morita equivalent to  $k\mathcal{C}_0$ , where  $\mathcal{C}_0$  is any skeleton of  $\mathcal{C}$ . So we may assume that  $\mathcal{C}$  is *skeletal*, that is, for any two distinct objects  $x$  and  $y$  in  $\mathcal{C}$ ,  $x$  is not isomorphic to  $y$ .

The category  $\mathcal{C}$  is called a *finite EI category* provided that all endomorphisms in  $\mathcal{C}$  are isomorphisms. In particular,  $\text{Hom}_{\mathcal{C}}(x, x) = \text{Aut}_{\mathcal{C}}(x)$  is a finite group for any object  $x$  in  $\mathcal{C}$ . Denote by  $k\text{Aut}_{\mathcal{C}}(x)$  the group algebra.

Throughout this paper, we assume that  $\mathcal{C}$  is a finite EI category which is skeletal, and  $\text{Obj}\mathcal{C} = \{x_1, x_2, \dots, x_n\}$ ,  $n \geq 2$ , satisfying  $\text{Hom}_{\mathcal{C}}(x_i, x_j) = \emptyset$  if  $i < j$ .

Let  $M_{ij} = k\text{Hom}_{\mathcal{C}}(x_j, x_i)$ . Write  $R_i = M_{ii}$ , which is a group algebra. Recall that the category algebra  $k\mathcal{C}$  is isomorphic to the corresponding upper triangular

$$\text{matrix algebra } \Gamma_{\mathcal{C}} = \begin{pmatrix} R_1 & M_{12} & \cdots & M_{1n} \\ & R_2 & \cdots & M_{2n} \\ & & \ddots & \vdots \\ & & & R_n \end{pmatrix}; \text{ see [5, Section 4].}$$

Let us fix a field  $k$ . Denote by  $k\text{-mod}$  the category of finite dimensional  $k$ -vector spaces and  $(k\text{-mod})^{\mathcal{C}}$  the category of covariant functors from  $\mathcal{C}$  to  $k\text{-mod}$ .

Recall that the category  $k\mathcal{C}\text{-mod}$  of left modules over the category algebra  $k\mathcal{C}$ , is identified with  $(k\text{-mod})^{\mathcal{C}}$ ; see [7, Proposition 2.1]. The category  $k\mathcal{C}\text{-mod}$  is a symmetric monoidal category. More precisely, the tensor product  $-\hat{\otimes}-$  is defined by

$$(M \hat{\otimes} N)(x) = M(x) \otimes_k N(x)$$

for any  $M, N \in (k\text{-mod})^{\mathcal{C}}$  and  $x \in \text{Obj}\mathcal{C}$ , and  $\alpha.(m \otimes n) = \alpha.m \otimes \alpha.n$  for any  $\alpha \in \text{Mor}\mathcal{C}$ ,  $m \in M(x)$ ,  $n \in N(x)$ ; see [8, 9].

Let  $\mathcal{C}$  be a finite EI category, and  $\Gamma = \Gamma_{\mathcal{C}} = \begin{pmatrix} R_1 & M_{12} & \cdots & M_{1n} \\ & R_2 & \cdots & M_{2n} \\ & & \ddots & \vdots \\ & & & R_n \end{pmatrix}$  be the corresponding upper triangular matrix algebra. Let  $X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$  and  $Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$  be two  $\Gamma$ -modules, where the left  $\Gamma$ -module structure maps are denoted by  $\varphi_{ij}^X$  and  $\varphi_{ij}^Y$ , respectively. We observe that  $X \hat{\otimes} Y = \begin{pmatrix} X_1 \otimes_k Y_1 \\ \vdots \\ X_n \otimes_k Y_n \end{pmatrix}$ , where the module

structure map  $\varphi_{ij} : M_{ij} \otimes_{R_j} (X_j \otimes_k Y_j) \rightarrow X_i \otimes_k Y_i$  is induced by the following:  $\varphi_{ij}(\alpha_{ij} \otimes (a_j \otimes b_j)) = \varphi_{ij}^X(\alpha_{ij} \otimes a_j) \otimes \varphi_{ij}^Y(\alpha_{ij} \otimes b_j)$ , where  $\alpha_{ij} \in \text{Hom}_{\mathcal{C}}(x_j, x_i)$ ,  $a_j \in X_j$  and  $b_j \in Y_j$ .

**Definition 3.1.** Let  $\mathcal{C}$  be a finite EI category, and  $\Gamma$  be the corresponding upper triangular matrix algebra. We say that  $\mathcal{C}$  is *GPT-closed*, if  $X, Y \in \Gamma\text{-Gproj}$  implies  $X \hat{\otimes} Y \in \Gamma\text{-Gproj}$ .

Recall that a finite EI category  $\mathcal{C}$  is *projective over  $k$*  if each  $k\text{Aut}_{\mathcal{C}}(y)\text{-}k\text{Aut}_{\mathcal{C}}(x)$ -bimodule  $k\text{Hom}_{\mathcal{C}}(x, y)$  is projective on both sides; see [5, Definition 4.2]. We recall the fact that the category algebra  $k\mathcal{C}$  is Gorenstein if and only if  $\mathcal{C}$  is projective over  $k$ ; see [5, Proposition 5.1].

Let  $\mathcal{C}$  be a finite projective EI category, and  $\Gamma$  be the corresponding upper triangular matrix algebra. Denote by  $C_i$  the  $i$ -th column of  $\Gamma$  which is a  $\Gamma\text{-}R_i$ -bimodule and projective on both sides.

**Proposition 3.2.** Let  $\mathcal{C}$  be a finite projective EI category, and  $\Gamma$  be the corresponding upper triangular matrix algebra. Then the following statements are equivalent.

- (1) The category  $\mathcal{C}$  is GPT-closed.
- (2) For any  $1 \leq p \leq q \leq n$ ,  $C_p \hat{\otimes} C_q \in \Gamma\text{-Gproj}$ .
- (3) For any  $1 \leq p \leq q \leq n$ ,  $C_p \hat{\otimes} C_q \in \Gamma\text{-proj}$ .

*Proof.* “(1) $\Rightarrow$ (2)” and “(3) $\Rightarrow$ (2)” are obvious.

“(2) $\Rightarrow$ (3)” We only need to prove that the  $\Gamma$ -module  $C_p \hat{\otimes} C_q$  has finite projective dimension, since a Gorenstein-projective module with finite projective dimension

is projective. We have  $C_p \hat{\otimes} C_q = \begin{pmatrix} M_{1p} \otimes_k M_{1q} \\ \vdots \\ M_{p-1,p} \otimes_k M_{p-1,q} \\ R_p \otimes_k M_{pq} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ . Since  $\mathcal{C}$  is projective, we

have that each  $M_{ip}$  is a projective  $R_i$ -module for  $1 \leq i \leq p$ . Then each  $M_{ip} \otimes_k M_{iq}$  is a projective  $R_i$ -module since  $R_i$  is a group algebra for  $1 \leq i \leq p$ . Hence the  $\Gamma$ -module  $C_p \hat{\otimes} C_q$  has finite projective dimension by [5, Corollary 3.6]. Then we are done.

“(2) $\Rightarrow$ (1)” We have that  $\Gamma$  is a Gorenstein algebra by [5, Proposition 5.1]. Then there is  $d \geq 0$  such that  $\Gamma$  is a  $d$ -Gorenstein algebra.

For any  $M \in \Gamma\text{-Gproj}$ , consider the following exact sequence

$$0 \rightarrow M \rightarrow P_0 \rightarrow P_1 \rightarrow \cdots \rightarrow P_d \rightarrow Y \rightarrow 0$$

with  $P_i$  projective,  $0 \leq i \leq d$ . Applying  $-\hat{\otimes} N$  on the above exact sequence, we have an exact sequence

$$0 \rightarrow M \hat{\otimes} N \rightarrow P_0 \hat{\otimes} N \rightarrow P_1 \hat{\otimes} N \rightarrow \cdots \rightarrow P_d \hat{\otimes} N \rightarrow Y \hat{\otimes} N \rightarrow 0, \quad (3.1)$$

since the tensor product  $-\hat{\otimes}-$  is exact in both variables. If  $N$  is projective, we have that each  $P_i \hat{\otimes} N$  is Gorenstein-projective for  $0 \leq i \leq d$  by (2). Then we have  $M \hat{\otimes} N \in \Gamma\text{-Gproj}$  by Lemma 2.3 (3). If  $N$  is Gorenstein-projective, we have that each  $P_i \hat{\otimes} N$  is Gorenstein-projective for  $0 \leq i \leq d$  in exact sequence (3.1) by the above process. Then we have  $M \hat{\otimes} N \in \Gamma\text{-Gproj}$  by Lemma 2.3 (3). Then we are done.  $\square$

The argument in “(2) $\Rightarrow$ (3)” of Proposition 3.2 implies the following result. It follows that the tensor product  $-\hat{\otimes}-$  on  $\Gamma\text{-Gproj}$  induces the one on  $\Gamma\text{-Gproj}$ , still denoted by  $-\hat{\otimes}-$ .

**Lemma 3.3.** *Assume that  $\mathcal{C}$  is GPT-closed. Let  $M \in \Gamma\text{-Gproj}$  and  $P \in \Gamma\text{-proj}$ . Then  $M \hat{\otimes} P \in \Gamma\text{-proj}$ .  $\square$*

Let  $\mathcal{C}$  be a finite projective EI category, and  $\Gamma$  be the corresponding upper triangular matrix algebra of  $\mathcal{C}$ . Recall that a complex in  $D^b(\Gamma\text{-mod})$ , the bounded derived category of finitely generated left  $\Gamma$ -modules, is called a *perfect complex* if it is isomorphic to a bounded complex of finitely generated projective modules. Recall from [2] that the *singularity category* of  $\Gamma$ , denoted by  $D_{sg}(\Gamma)$ , is the Verdier quotient category  $D^b(\Gamma\text{-mod})/\text{perf}(\Gamma)$ , where  $\text{perf}(\Gamma)$  is a thick subcategory of  $D^b(\Gamma\text{-mod})$  consisting of all perfect complexes.

Recall from [6] that there is a triangle equivalence

$$F : \Gamma\text{-Gproj} \xrightarrow{\sim} D_{sg}(\Gamma) \quad (3.2)$$

sending a Gorenstein-projective module to the corresponding stalk complex concentrated on degree zero. The functor  $F$  transports the tensor product on  $D_{sg}(\Gamma)$  to  $\Gamma\text{-Gproj}$  such that the category  $\Gamma\text{-Gproj}$  becomes a tensor triangulated category.

**Proposition 3.4.** *Let  $\mathcal{C}$  be a finite projective EI category, and  $\Gamma$  be the corresponding upper triangular matrix algebra of  $\mathcal{C}$ . Assume that the category  $\mathcal{C}$  is GPT-closed. Then the tensor product  $-\hat{\otimes}-$  on  $\Gamma\text{-Gproj}$  induced by the tensor product on  $\Gamma\text{-Gproj}$  coincide with the one transported from  $D_{sg}(\Gamma)$ , up to natural isomorphism.*

*Proof.* Consider the functor  $F$  in (3.2). Recall that the tensor product on  $D_{sg}(\Gamma)$  is induced by the tensor product  $-\hat{\otimes}-$  on  $D^b(\Gamma\text{-mod})$ , where the later is given by  $-\hat{\otimes}-$  on  $\Gamma\text{-mod}$ . We have  $F(M) \hat{\otimes} F(N) = F(M \hat{\otimes} N)$  in  $D_{sg}(\Gamma)$  for any  $M, N \in \Gamma\text{-Gproj}$ . This implies that  $F$  is a tensor triangle equivalence. Then we are done.  $\square$

Let  $k$  be a field and  $G$  be a finite group. Recall that a left (resp. right)  $G$ -set is a set with a left (resp. right)  $G$ -action. Let  $Y$  be a left  $G$ -set and  $X$  be a right  $G$ -set. Recall an equivalence relation “ $\sim$ ” on the product  $X \times Y$  as follows:  $(x, y) \sim (x', y')$  if and only if there is an element  $g \in G$  such that  $x = x'g$  and  $y = g^{-1}y'$  for  $x, x' \in X$  and  $y, y' \in Y$ . Write the quotient set  $X \times Y / \sim$  as  $X \times_G Y$ .

The following two lemmas are well known.

**Lemma 3.5.** *Let  $Y$  be a left  $G$ -set and  $X$  be a right  $G$ -set. Then there is an isomorphism of  $k$ -vector spaces*

$$\varphi : kX \otimes_{kG} kY \xrightarrow{\sim} k(X \times_G Y), \quad x \otimes y \mapsto (x, y),$$

where  $x \in X$  and  $y \in Y$ .  $\square$

**Lemma 3.6.** *Let  $Y_1$  and  $Y_2$  be two left  $G$ -sets. Then we have an isomorphism of left  $kG$ -modules*

$$\varphi : kY_1 \otimes_k kY_2 \xrightarrow{\sim} k(Y_1 \times Y_2), \quad y_1 \otimes y_2 \mapsto (y_1, y_2),$$

where  $y_1 \in Y_1, y_2 \in Y_2$ .  $\square$

**Lemma 3.7.** *Let  $\mathcal{C}$  be a finite projective EI category, and  $\Gamma$  be the corresponding upper triangular matrix algebra. Assume  $1 \leq p \leq q \leq n$ . Then  $C_p \hat{\otimes} C_q \in \Gamma\text{-proj}$  implies that each morphism in  $\bigsqcup_{y \in \text{Obj}\mathcal{C}} \text{Hom}_{\mathcal{C}}(x_p, y)$  is a monomorphism.*



*Proof.* We have  $C_p \hat{\otimes} C_q = \begin{pmatrix} M_{1p} \otimes_k M_{1q} \\ \vdots \\ M_{p-1,p} \otimes_k M_{p-1,q} \\ R_p \otimes_k M_{pq} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ . Then each  $R_i$ -map

$$\varphi_{ip} : M_{ip} \otimes_{R_p} (R_p \otimes_k M_{pq}) \rightarrow M_{ip} \otimes_k M_{iq}$$

sending  $\alpha \otimes (g \otimes \beta)$  to  $\alpha \circ g \otimes \alpha \otimes \beta$ , where  $\alpha \in \text{Hom}_{\mathcal{C}}(x_p, x_i), g \in \text{Aut}_{\mathcal{C}}(x_p), \beta \in \text{Hom}_{\mathcal{C}}(x_q, x_p)$ , is injective for  $1 \leq i < p \leq q \leq n$  by Corollary 2.6. We have that the sets  $\text{Hom}_{\mathcal{C}}(x_p, x_i) \times_{\text{Aut}_{\mathcal{C}}(x_p)} (\text{Aut}_{\mathcal{C}}(x_p) \times \text{Hom}_{\mathcal{C}}(x_q, x_p))$  and  $\text{Hom}_{\mathcal{C}}(x_p, x_i) \times \text{Hom}_{\mathcal{C}}(x_q, x_i)$  are  $k$ -basis of  $M_{ip} \otimes_{R_p} (R_p \otimes_k M_{pq})$  and  $M_{ip} \otimes_k M_{iq}$ , respectively by Lemma 3.5 and Lemma 3.6. For each  $1 \leq i < p$ , since  $\varphi_{ip}$  is injective, we have an injective map

$$\varphi : \text{Hom}_{\mathcal{C}}(x_p, x_i) \times_{\text{Aut}_{\mathcal{C}}(x_p)} (\text{Aut}_{\mathcal{C}}(x_p) \times \text{Hom}_{\mathcal{C}}(x_q, x_p)) \rightarrow \text{Hom}_{\mathcal{C}}(x_p, x_i) \times \text{Hom}_{\mathcal{C}}(x_q, x_i)$$

sending  $(\alpha, (g, \beta))$  to  $(\alpha \circ g, \alpha \otimes \beta)$ , for  $\alpha \in \text{Hom}_{\mathcal{C}}(x_p, x_i), g \in \text{Aut}_{\mathcal{C}}(x_p), \beta \in \text{Hom}_{\mathcal{C}}(x_q, x_p)$ .

For each  $1 \leq i < p$ , and  $\alpha \in \text{Hom}_{\mathcal{C}}(x_p, x_i)$ , let  $\beta, \beta' \in \text{Hom}_{\mathcal{C}}(x_q, x_p)$  satisfy  $\alpha \otimes \beta = \alpha \otimes \beta'$ . Then we have  $(\alpha, \alpha \otimes \beta) = (\alpha, \alpha \otimes \beta')$ , that is,  $\varphi(\alpha, (\text{Id}_{x_p}, \beta)) = \varphi(\alpha, (\text{Id}_{x_p}, \beta'))$ . Since  $\varphi$  is injective, we have  $(\alpha, (\text{Id}_{x_p}, \beta)) = (\alpha, (\text{Id}_{x_p}, \beta'))$  in  $\text{Hom}_{\mathcal{C}}(x_p, x_i) \times_{\text{Aut}_{\mathcal{C}}(x_p)} (\text{Aut}_{\mathcal{C}}(x_p) \times \text{Hom}_{\mathcal{C}}(x_q, x_p))$ . Hence  $\beta = \beta'$ . Then we have that  $\alpha$  is a monomorphism.  $\square$

**Proposition 3.8.** *Let  $\mathcal{C}$  be a finite projective EI category. Assume that  $\mathcal{C}$  is GPT-closed. Then that each morphism in  $\mathcal{C}$  is a monomorphism.*

*Proof.* It follows from Proposition 3.2 and Lemma 3.7.  $\square$

Let  $\mathcal{P}$  be a finite poset. We assume that  $\text{Obj}\mathcal{P} = \{x_1, \dots, x_n\}$  satisfying  $x_i \not\leq x_j$  if  $i < j$ , and  $\Gamma$  is the corresponding upper triangular matrix algebra. We observe that each entry of  $\Gamma$  is 0 or  $k$ , and each projective  $\Gamma$ -module is a direct sum of some  $C_i$  for  $1 \leq i \leq n$ . For any  $a, b \in \text{Obj}\mathcal{P}$  satisfying  $a \not\leq b$  and  $b \not\leq a$ , denote by  $L_{a,b} = \{x \in \text{Obj}\mathcal{P} \mid a < x, b < x\}$ .

**Example 3.9.** Let  $\mathcal{P}$  be a finite poset. Then  $\mathcal{P}$  is GPT-closed if and only if any two distinct minimal elements in  $L_{a,b}$  has no common upper bound for  $a, b \in \text{Obj}\mathcal{P}$  satisfying  $a \not\leq b$  and  $b \not\leq a$ .

For the “if” part, assume that any two distinct minimal elements in  $L_{a,b}$  has no common upper bound. By Proposition 3.2, we only need to prove that  $C_t \hat{\otimes} C_n$  is projective for  $1 \leq t \leq n$ , since the general case of  $C_t \hat{\otimes} C_j$  can be considered in  $\Gamma_{\max\{t,j\}}$ .

For each  $1 \leq t \leq n$ , if  $(C_n)_t = k$ , that is,  $x_n \leq x_t$ , then  $(C_t)_i = k$  implies  $(C_n)_i = k$  for  $1 \leq i \leq t$ . Hence we have  $C_t \hat{\otimes} C_n \simeq C_t$ . Assume that  $(C_n)_t = 0$ , that is,  $x_n \not\leq x_t$ . Let  $L'_{x_t, x_n} = \{x_{s_1}, \dots, x_{s_r}\}$  be all distinct minimal elements in  $L_{x_t, x_n}$ . For each  $1 \leq i < t$ , if  $(C_t)_i = k = (C_n)_i$ , that is,  $x_n \leq x_i, x_t \leq x_i$ , then there is a unique  $x_{s_l} \in L'_{x_t, x_n}$  satisfying  $x_{s_l} \leq x_i$ , that is, there is a unique  $x_{s_l} \in L'_{x_t, x_n}$  satisfying  $(C_{s_l})_i = k$ , since any two distinct elements in  $L'_{x_t, x_n}$  has no common upper bound. Then we have  $C_t \hat{\otimes} C_n \simeq \bigoplus_{l=1}^r C_{s_l}$ .



For the “only if” part, assume that  $x_t, x_j \in \text{Obj}\mathcal{P}$  satisfying  $x_t \not\leq x_j$  and  $x_j \not\leq x_t$  and  $C_t \hat{\otimes} C_j \simeq \bigoplus_{l=1}^r C_{s_l}$ . Then each  $x_{s_l} \in L_{x_t, x_j}$ . Assume that  $x_{s_1}$  and  $x_{s_2}$  be two distinct minimal elements in  $L_{x_t, x_j}$  having a common upper bound  $x_i$ . Then  $(C_t \hat{\otimes} C_j)_i = k$  and  $(C_{s_1} \oplus C_{s_2})_i = k \oplus k$ , which is a contradiction.  $\square$

#### 4. PROOF OF THEOREM 1.2

Let  $\mathcal{C}$  be a finite EI category. Recall from [3, Definition 2.3] that a morphism  $x \xrightarrow{\alpha} y$  in  $\mathcal{C}$  is *unfactorizable* if  $\alpha$  is not an isomorphism and whenever it has a factorization as a composite  $x \xrightarrow{\beta} z \xrightarrow{\gamma} y$ , then either  $\beta$  or  $\gamma$  is an isomorphism. Let  $x \xrightarrow{\alpha} y$  in  $\mathcal{C}$  be an unfactorizable morphism. Then  $h \circ \alpha \circ g$  is also unfactorizable for every  $h \in \text{Aut}_{\mathcal{C}}(y)$  and every  $g \in \text{Aut}_{\mathcal{C}}(x)$ ; see [3, Proposition 2.5]. Let  $x \xrightarrow{\alpha} y$  in  $\mathcal{C}$  be a morphism with  $x \neq y$ . Then it has a decomposition  $x = x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} x_n = y$  with all  $\alpha_i$  unfactorizable; see [3, Proposition 2.6].

Following [3, Definition 2.7], we say that a finite EI category  $\mathcal{C}$  satisfies the Unique Factorization Property (UFP), if whenever a non-isomorphism  $\alpha$  has two decompositions into unfactorizable morphisms:

$$x = x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} x_m = y$$

and

$$x = y_0 \xrightarrow{\beta_1} y_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_n} y_n = y,$$

then  $m = n$ ,  $x_i = y_i$ , and there are  $h_i \in \text{Aut}_{\mathcal{C}}(x_i)$ ,  $1 \leq i \leq m - 1$ , such that the following diagram commutes :

$$\begin{array}{ccccccccccc} x = x_0 & \xrightarrow{\alpha_1} & x_1 & \xrightarrow{\alpha_2} & x_2 & \xrightarrow{\alpha_3} & \cdots & \xrightarrow{\alpha_{m-1}} & x_{m-1} & \xrightarrow{\alpha_m} & x_m = y \\ \parallel & & \downarrow h_1 & & \downarrow h_2 & & & & \downarrow h_{m-1} & & \parallel \\ x = x_0 & \xrightarrow{\beta_1} & x_1 & \xrightarrow{\beta_2} & x_2 & \xrightarrow{\beta_3} & \cdots & \xrightarrow{\beta_{m-1}} & x_{m-1} & \xrightarrow{\beta_m} & x_m = y \end{array}$$

Let  $\mathcal{C}$  be a finite EI category. Following [4, Section 6], we say that  $\mathcal{C}$  is a finite *free* EI category if it satisfies the UFP. By [3, Proposition 2.8], this is equivalent to the original definition [3, Definition 2.2].

Let  $n \geq 2$ . Let  $\mathcal{C}$  be a finite projective and free EI category with  $\text{Obj}\mathcal{C} = \{x_1, x_2, \dots, x_n\}$  satisfying  $\text{Hom}_{\mathcal{C}}(x_i, x_j) = \emptyset$  if  $i < j$  and  $\Gamma$  be the corresponding upper triangular matrix algebra of  $\mathcal{C}$ . Then  $\Gamma$  is 1-Gorenstein; see [5, Theorem 5.3].

Set  $\text{Hom}_{\mathcal{C}}^0(x_j, x_i) = \{\alpha \in \text{Hom}_{\mathcal{C}}(x_j, x_i) \mid \alpha \text{ is unfactorizable}\}$ . Denote by  $M_{ij}^0 = k\text{Hom}_{\mathcal{C}}^0(x_j, x_i)$ , which is an  $R_i$ - $R_j$ -sub-bimodule of  $M_{ij}$ ; see [5, Notation 4.8]. Recall the left  $\Gamma_t$ -module  $M_t^*$  and the right  $\Gamma'_{n-t}$ -module  $M_t^{**}$  in Notation 2.1, for  $1 \leq t \leq n - 1$ . Observe that  $M_t^{**} \simeq (M_{t,t+1}^0, M_{t,t+2}^0, \dots, M_{tn}^0) \otimes_{\Gamma'_{D,n-t}} \Gamma'_{n-t}$ ; compare [5,

Lemmas 4.10 and 4.11], which implies that  $M_t^* \simeq \Gamma_t \otimes_{\Gamma_t^p} \begin{pmatrix} M_{1,t+1}^0 \\ \vdots \\ M_{t,t+1}^0 \end{pmatrix}$ .

Let  $X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$  be a left  $\Gamma$ -module. For each  $1 \leq t \leq n-1$ , we have

$$\begin{aligned} M_t^{**} \otimes_{\Gamma'_{n-t}} \begin{pmatrix} X_{t+1} \\ \vdots \\ X_n \end{pmatrix} &\simeq (M_{t,t+1}^0, M_{t,t+2}^0, \dots, M_{tn}^0) \otimes_{\Gamma'_{D,n-t}} \Gamma'_{n-t} \otimes_{\Gamma'_{n-t}} \begin{pmatrix} X_{t+1} \\ \vdots \\ X_n \end{pmatrix} \\ &\simeq (M_{t,t+1}^0, M_{t,t+2}^0, \dots, M_{tn}^0) \otimes_{\Gamma'_{D,n-t}} \begin{pmatrix} X_{t+1} \\ \vdots \\ X_n \end{pmatrix} \\ &\simeq \bigoplus_{j=t+1}^n M_{tj}^0 \otimes_{R_j} X_j. \end{aligned}$$

Recall the  $R_t$ -map  $\varphi_t^{**}$  in Lemma 2.5. Here, we observe that

$$\varphi_t^{**} : \bigoplus_{j=t+1}^n M_{tj}^0 \otimes_{R_j} X_j \rightarrow X_t, \quad \sum_{j=t+1}^n (m_j \otimes x_j) \mapsto \sum_{j=t+1}^n \varphi_{tj}(m_j \otimes x_j).$$

**Lemma 4.1.** *Let  $\mathcal{C}$  be a finite projective and free EI category and  $\Gamma$  be the corresponding upper triangular matrix algebra. Assume  $1 \leq p \leq q \leq n$ . If each morphism in  $\bigsqcup_{y \in \text{Obj } \mathcal{C}} \bigsqcup_{j=1}^p \text{Hom}_{\mathcal{C}}(x_j, y)$  is a monomorphism, then  $C_p \hat{\otimes} C_q \in \Gamma\text{-proj}$ .*

*Proof.* We only need to prove that each  $R_t$ -map

$$\varphi_t^{**} : \bigoplus_{j=t+1}^p M_{tj}^0 \otimes_{R_j} (M_{jp} \otimes_k M_{jq}) \rightarrow M_{tp} \otimes_k M_{tq}$$

is injective for  $1 \leq t < p$  by Lemma 2.5 and Proposition 3.2.

By Lemmas 3.5 and 3.6, we have that the set  $\text{Hom}_{\mathcal{C}}(x_p, x_t) \times \text{Hom}_{\mathcal{C}}(x_q, x_t)$  is a  $k$ -basis of  $M_{tp} \otimes_k M_{tq}$ , and the set

$$\bigsqcup_{j=t+1}^p \text{Hom}_{\mathcal{C}}^0(x_j, x_t) \times_{\text{Aut}_{\mathcal{C}}(x_j)} (\text{Hom}_{\mathcal{C}}(x_p, x_j) \times \text{Hom}_{\mathcal{C}}(x_q, x_j)) =: B$$

is a  $k$ -basis of  $\bigoplus_{j=t+1}^p M_{tj}^0 \otimes_{R_j} (M_{jp} \otimes_k M_{jq})$ .

We have the following commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\subseteq} & \bigoplus_{j=t+1}^p M_{tj}^0 \otimes_{R_j} (M_{jp} \otimes_k M_{jq}) \\ \downarrow \varphi_t^{**}|_B & & \downarrow \varphi_t^{**} \\ \text{Hom}_{\mathcal{C}}(x_p, x_t) \times \text{Hom}_{\mathcal{C}}(x_q, x_t) & \xrightarrow{\subseteq} & M_{tp} \otimes_k M_{tq} \end{array}$$

Observe that  $\varphi_t^{**}$  is injective if and only if  $\varphi_t^{**}|_B$  is injective for each  $1 \leq t < p$ .

Assume that  $\varphi_t^{**}(\alpha, (\beta, \theta)) = \varphi_t^{**}(\alpha', (\beta', \theta'))$ , where  $\alpha \in \text{Hom}_{\mathcal{C}}^0(x_j, x_t)$ ,  $\beta \in \text{Hom}_{\mathcal{C}}(x_p, x_j)$ ,  $\theta \in \text{Hom}_{\mathcal{C}}(x_q, x_j)$  and  $\alpha' \in \text{Hom}_{\mathcal{C}}^0(x_{j'}, x_t)$ ,  $\beta' \in \text{Hom}_{\mathcal{C}}(x_p, x_{j'})$ ,  $\theta' \in \text{Hom}_{\mathcal{C}}(x_q, x_{j'})$ . Then we have  $\alpha\beta = \alpha'\beta'$  in  $\text{Hom}_{\mathcal{C}}(x_p, x_t)$  and  $\alpha\theta = \alpha'\theta'$  in  $\text{Hom}_{\mathcal{C}}(x_q, x_t)$ . Since  $\mathcal{C}$  is free and  $\alpha, \alpha'$  are unfactorizable, we have that  $j = j'$  and there is  $g \in \text{Aut}_{\mathcal{C}}(x_j)$  such that  $\alpha = \alpha'g$  and  $\beta = g^{-1}\beta'$ . Since  $\alpha\theta = \alpha'\theta' =$

$\alpha g^{-1}\theta'$  and  $\alpha$  is a monomorphism, we have that  $\theta = g^{-1}\theta'$ . Then we have that  $(\alpha, (\beta, \theta)) = (\alpha'g, (g^{-1}\beta', g^{-1}\theta')) = (\alpha', (\beta', \theta'))$ , which implies that the map  $\varphi_t^{**}|_B$  is injective.  $\square$

**Theorem 4.2.** *Let  $\mathcal{C}$  be a finite projective and free EI category. Then the category  $\mathcal{C}$  is GPT-closed if and only if each morphism in  $\mathcal{C}$  is a monomorphism.*

*Proof.* The “if” part is just by Proposition 3.2 and Lemma 4.1. The “only if” part is just by Proposition 3.8.  $\square$

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